SIMULTANEOUS EXTENSIONS OF VECTOR MEASURES

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The literature concerning the extension of vector measures is very extensive, the most significant results known until now being those presented in [1], [2], [3], [4], [6], [7], [10], [11].

The aim of our paper is to prove that the s-bounded measures defined on a Boolean algebra $\mathcal{C}$ can be extended simultaneously to any Boolean algebra $\tilde{\mathcal{C}} \supset \mathcal{C}$. More precisely, we show that for each Banach space E and for each finite additive measure $\mu : \tilde{\mathcal{C}} \to \mathbf{R}_+$ there exists a linear mapping which associates to each s-bounded measure $m : \mathcal{C} \to E$, $m \ll \mu|\tilde{\mathcal{C}}$, an s-bounded measure $\tilde{m} : \mathcal{C} \to E$ such that $\tilde{m}|\mathcal{C} = m$ and $\tilde{m} \ll \mu$. It was remarked in [2] that the Lebesgue measure on [0, 1] cannot be extended as a σ-additive measure to the σ-algebra of all subsets of [0, 1], and thus our result is the best possible.

The key point in our proof is the use of the geometric properties of AM and AL-spaces (in the sense of Kakutani) and Dinculeanu's approach on vector measures as continuous linear mappings on the vector space of all totally measurable bounded functions.

Our main result unables us to prove in a unifying manner previous theorems on extension of vector measures such as Dinculeanu's, Kluvanek's, etc.

1. REVIEW ON VECTOR MEASURES

First recall the construction of the Banach lattice $\mathcal{M}(\mathcal{C})$ of all „bounded real functions" for $\mathcal{C}$ a Boolean algebra. For, consider the set of all elements of the form $\sum\limits_{i \in F} \alpha_i A_i$ where

$$A_i \in \mathcal{C},\ A_i \neq 0,\ \inf(A_i, A_j) = 0 \text{ for } i \neq j,$$

$\sup\{A_i ;\ i \in F\} = T$, F is an arbitrary finite set and T is the greatest element of $\mathcal{C}$. Say $\sum\limits_{i \in F} \alpha_i A_i \sim \sum\limits_{j \in G} \beta_j B_j$ iff $\inf(A_i, B_j) \neq 0$ implies

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$\alpha_i = \beta_j$. The set $\mathcal{E}(\mathcal{C})$ of all classes of equivalence as obtained can be endowed with a structure of normed lattice as follows :

$$\widehat{\Sigma\, \alpha_i A_i} + \widehat{\Sigma \beta_j B_j} = \overline{\Sigma(\alpha_i + \beta_j) \inf\, (A_i, B_i)}$$

$$\lambda \widehat{\Sigma \alpha_i A_i} = \widehat{\Sigma \lambda \alpha_i A_i},\ \lambda \in \mathbf{R}$$

$$\widehat{\Sigma \alpha_i A_i} \geqslant 0 \text{ if, and only if, all } \alpha_i \text{ are positive}$$

$$||\widehat{\Sigma \alpha_i A_i}|| = \sup|\alpha_i|.$$

Then $\mathcal{M}(\mathcal{C})$ can be defined as the topological completion of $\mathcal{E}(\mathcal{C})$. Clearly $\mathcal{M}(\mathcal{C})$ is an AM-space with unit (in the sense of Kakutani) and the classical result (due to M.H. Stone) concerning the concrete representation of Boolean algebras yields that $\mathcal{M}(\mathcal{C})$ is lattice isometric to $C(S)$, for S the spectrum of $\mathcal{C}$.

For E a sequentially complete locally convex space and $\mathcal{C}$ a Boolean algebra we shall denote by $\mathrm{Mes}_E(\mathcal{C})$ the vector space of all (finitely additive) measures $m : \mathcal{C} \to E$ such that

$$\sup_{A \in \mathcal{C}} q(m(A)) < \infty$$

for each continuous semi-norm q on E.

A measure $m \in Mes_E(\mathcal{C})$ is called s-bounded if for every sequence of pairwise disjoint elements $A_n \in \mathcal{C}$ we have $\lim m(A_n) = 0$. The vector subspace of all s-bounded measures $m \in Mes_E(\mathcal{C})$ will be denoted by $M_E(\mathcal{C})$.

Given a positive measure $\mu \in M_{\mathbf{R}}(\mathcal{C})$, we shall denote by $M_E(\mathcal{C}, \mu)$ the vector subspace of all $m \in M_E(\mathcal{C})$ such that $m \ll \mu$ i.e., $\lim\limits_{\mu(A)\to 0} m(A) = 0$.

The following result which establishes an equivalence between operators and measures goes back to Dinculeanu [5] :

1.1. **Theorem.** *There exists a natural algebraic isomorphism*

$$\Phi_{\mathcal{C}, E} : Mes_E(\mathcal{C}) \to \mathcal{L}(\mathcal{M}(\mathcal{C}), E)$$

given by :

$$\Phi_{\mathcal{C}, E}(m)(\chi_A) = m(A)$$

for every $A \in \mathcal{C}$. *Here* χ_A *denotes the class of* $1 \cdot A + 0 \cdot A^{\perp}$ *where* $A^{\perp}$ *is the unique element of* $\mathcal{C}$ *such that sup* $(A, A^{\perp}) = T$ *and inf* $(A, A^{\perp}) = 0$.

The above isomorphism can be precised as follows :

I) *If* E *is an ordered locally convex space then* $\Phi_{\mathcal{C}, E}$ *is order preserving.*

II) (*J. Hoffmann-Jorgensen* [9]). *By the isomorphism* $\Phi_{\mathcal{C}, E}$ *the weakly compact operators of* $\mathcal{L}(\mathcal{M}(\mathcal{C}), E)$ *correspond precisely to the s-bounded measures of* $Mes_E(\mathcal{C})$.

III) (*C. Niculescu* [15]) *If* $\mu : \mathcal{C} \to \mathbf{R}$ *is a positive measure then* $m \in M_E(\mathcal{C}, \mu)$ *iff* $\Phi_{\mathcal{C},E}(m)$ *is absolutely continuous with respect to* μ i.e.,

$$||\Phi_{\mathcal{C},E}(m)(f)|| \leqslant \varepsilon||f|| + \delta(\varepsilon)\int|f|\,d\mu$$

for every $f \in \mathcal{M}(\mathcal{C})$ *and every* $\varepsilon > 0$. *Here*

$$\int h\,d\mu = \Phi_{\mathcal{C},\mathbf{R}}(\mu)(h).$$

Sketch of the proof. The non trivial assertions are II) and III).

II). Let us denote by S the spectrum of $\mathcal{C}$. Then S is a compact Hausdorff space and $\mathcal{C}$ is isomorphic to the Boolean algebra of all clopen subsets of S, which implies that $\mathcal{M}(\mathcal{C})$ is lattice isometric to $C(S)$. Notice also that every open F_σ subset of S is of the form $D = \bigcup K_n$ for $\{K_n\}_n$ a suitable increasing sequence of clopen subsets of S. By Lebesgue's theorem on dominated convergence it follows that $\chi_{K_n} \to \chi_D$ in the $\sigma(C(S)^{**}, C(S)^*)$ topology of $C(S)^{**}$.

Let $m \in M_E(\mathcal{C})$. Then $\Phi_{\mathcal{C},E}(m)(\chi_{K_n}) = m(K_n)$ is a converging sequence and thus $\Phi_{\mathcal{C},E}(m)^{**}(\chi_D) = m(D) \in E$. By Theorem 6 in [8] we obtain that $\Phi_{\mathcal{C},E}(m)$ is weakly compact as an operator defined on $C(S)$.

Conversely, if $\Phi_{\mathcal{C},E}(m)$ is weakly compact and $\{A_n\}_n$ is a sequence of pairwise disjoint elements of $\mathcal{C}$ then $\chi_{A_n}(s) \to 0$ for all $s \in S$ and Theorem 6 in [8[yields that the sequence $\Phi_{\mathcal{C},E}(m)(\chi_{A_n}) = m(A_n)$ is norm converging to 0, q.e.d.

III) Let $m \in M_E(\mathcal{C}, \mu)$. By (II), the set $\mathcal{K} = \{x^* \circ \Phi_{\mathcal{C},E}(m) ; x^* \in E^*, ||x^*|| \leqslant 1\}$ is weakly relatively compact and the classical criterion (due to Dunford and Pettits) of weak compactness in a space $L_1(\lambda)$ yields the same for $|\mathcal{K}| = \{|\nu| ; \nu \in \mathcal{K}\}$. It is convenient here to identify $\mathcal{M}(\mathcal{C})$ with $C(S)$, where S denotes the spectrum of $\mathcal{C}$, and to regard the functionals on $\mathcal{M}(\mathcal{C})$ as Radon measures on S. We shall prove the following estimate :

$$(*)\ \sup_{\nu\in|\mathcal{K}|} \int |f|d\nu \leqslant \varepsilon||f|| + \delta(\varepsilon)\int |f|d\mu,\ f\in C(S).$$

which implies III).

Indeed, if $(*)$ fails, then there exist a positive ε, a sequence $0 \leqslant f_n \leqslant 1$ in $C(S)$ and a sequence $\nu_n \in |\mathcal{K}|$ such that

$$\int f_n d\mu \leqslant 2^{-n-1} \text{ and } \int f_n d\nu_n \geqslant \varepsilon$$

for all $n \geqslant 1$. Put :

$$h_n(s) = \sup\{f_k(s) ; k \geqslant n\}$$

$$h(s) = \inf\{h_n(s) ; n \geqslant 1\}$$

Then h_n and h are Borel measurable functions and Theorem 2 in [8] implies that

$$\lim_{n\to\infty} \int h_n d\nu_k = \int h d\nu_k,$$

uniformly for $k \geqslant 1$.

Since $\nu_k \ll \mu$ and $\int h\,d\mu = 0$, we have $\int h\,d\nu_k = 0\ (k \in \mathrm{N})$ in contradiction with the fact that $\int h_n d\nu_n \geqslant \varepsilon\ (n \in \mathrm{N})$ and thus the estimate $(*)$ holds. The remainder of the proof is now clear.

2. SIMULTANEOUS EXTENSIONS OF S-BOUNDED MEASURES

Let $\mathcal{C} \subset \tilde{\mathcal{C}}$ two Boolean algebras.

2.1. Theorem.

I) If E is a sequentially complete locally convex space then there exists a linear mapping $e : M_E(\mathcal{C}) \to M_E(\tilde{\mathcal{C}})$ such that $e(m)|\mathcal{C} = m$ for every $m \in M_E(\mathcal{C})$.

II) If E is a Banach space and $\tilde{\mu} : \tilde{\mathcal{C}} \to \mathbf{R}$ is a positive measure then there exists a linear mapping $e : M_E(\mathcal{C}, \tilde{\mu}|\mathcal{C}) \to M_E(\tilde{\mathcal{C}}, \tilde{\mu})$ such that $e(m)|\mathcal{C} = m$ for every $m \in M_E(\mathcal{C}, \tilde{\mu}|\mathcal{C})$.

In both cases if E is an ordered locally convex space, e can be chosen to be positive.

Proof. Since $\mathcal{M}(\mathcal{C})$ is lattice isometric to a space $C(S)$, for S the spectrum of $\mathcal{C}$, the Banach lattice $\mathcal{M}(\mathcal{C})$ has the extension property and thus there exists a norm-1 positive projection $P : \mathcal{M}(\tilde{\mathcal{C}})^{**} \to \mathcal{M}(\mathcal{C})^{**}$. See [12], page 81. By using II) and III) in Theorem 1.1 above it follows that e can be defined by

$$e(m) = \Phi_{\tilde{\mathcal{C}}, E}^{-1}(\Phi_{\mathcal{C}, E}(m)^{**} \circ P|\mathcal{M}(\tilde{\mathcal{C}}))$$

q.e.d.

We have a unique extension if $\tilde{\mathcal{C}}$ is the Boolean σ-algebra generated by $\mathcal{C}$. This fact may be deduced from [10] but we prefer here a direct argument.

2.2. Proposition. *Each $m \in M_E(\mathcal{C})$ can be extended uniquely to the Boolean σ-algebra $\mathcal{T}$ generated by $\mathcal{C}$, as an s-bounded measure. In other words the canonical restriction $r : M_E(\mathcal{T}) \to M_E(\mathcal{C})$ is an isomorphism whose inverse is e.*

Proof. If S denotes the spectrum of $\mathcal{C}$ then $\mathcal{M}(\mathcal{C})$ is lattice isometric to $C(S)$ and $\Phi_{\mathcal{C}, E}(m)$ can be regarded as a weakly compact operator defined on $C(S)$. See Theorem 1.1 above. The operator $\Phi_{\mathcal{C}, E}(m)$ has a unique weakly compact extension to $\mathcal{M}(\mathcal{C})^{**} = C(S)^{**}$, particularly to $\mathcal{M}(\mathcal{B}(S))$, where $\mathcal{B}(S)$ denotes the Borel σ-algebra associated to S. By Stone's representation theorem $\mathcal{C}$ is isomorphic to the Boolean algebra $\mathcal{C}'$ of all clopen subsets of S. Clearly $\mathcal{C}' \subset \mathcal{B}(S)$ and thus m has a unique s-bounded extension to $\mathcal{B}(S)$ (use Theorem 1.1 (II) above) and a fortiori to the Boolean σ-algebra generated by $\mathcal{C}$, q.e.d.

2.3. Corollary (see [10]). *Every σ-additive measure $m \in M_E(\mathcal{C})$ has a unique σ-additive extension to the Boolean σ-algebra $\mathcal{T}$ generated by $\mathcal{C}$.*

Proof. By Proposition 2.2 above m has a unique extension $\tilde{m} \in M_E(\mathcal{T})$. Then $x^* \circ \tilde{m}$ is the unique s-bounded extension of $x^* \circ m$ to $\mathcal{T}$ whenever $x^* \in E^*$, A well known result due to Hahn (see [14], page 136) asserts that every σ-additive measure $\lambda : \mathcal{C} \to \mathbf{R}$ has a unique σ-additive extension to $\mathcal{T}$. Every σ-additive measure given on a Boolean σ-algebra is s-bounded and thus m is weakly σ-additive. It remains to apply a classical result due to Orlicz and Pettis in order to obtain the σ-additivity of $\tilde{m}$, q.e.d.

2.4. **Remark.** If E is a Banach space which contains no isomorphic copy of c_0 then $Mes_E(\mathcal{C}) = M_E(\mathcal{C})$ for every Boolean algebra $\mathcal{C}$. This well known fact can be deduced from [8] by using Theorem 1.1 above.

3. SIMULTANEOUS EXTENSIONS OF MAJORIZED MEASURES

For $\tilde{\mu} : \tilde{\mathcal{C}} \to \mathbf{R}$ a positive measure and E a Banach space we shall denote by $M_E(\tilde{\mu})$ the vector space of all $m \in M_E(\tilde{\mu})$ such that

$$\sup_{\tilde{\mu}(A) \neq 0} ||m(A)||/\tilde{\mu}(A) < \infty$$

If $\mathcal{C} \subset \tilde{\mathcal{C}}$ is another Boolean algebra then there is defined a linear mapping

$$r : M_E(\tilde{\mu}) \to M_E(\tilde{\mu}|\mathcal{C})$$

given by the restriction of every $\tilde{m} \in M_E(\tilde{\mathcal{C}})$ to $\mathcal{C}$.

3.1. **Theorem.** *There exists a linear mapping*

$$e : M_E(\tilde{\mu}\ |\mathcal{C}) \to M_E(\tilde{\mu})$$

such that $r \circ e$ is the identity of $M_E(\tilde{\mu}|\mathcal{C})$.

As follows from Proposition 2.2 above, e is an isomorphism if $\tilde{\mathcal{C}}$ is the Boolean σ-algebra generated by $\mathcal{C}$.

Proof. We first describe a canonical method to construct AL spaces in the sense of Kakutani. Let X be *a* normed lattice i.e., $|x| \leqslant |y|$ implies $||x|| \leqslant ||y||$. Each $x^* \in X^*$, $x^* > 0$, defines on X a relation of equivalence as follows:

$$x \sim y \text{ if, and only if } x^*(|x - y|) = 0$$

The completion of $X/\sim$ with respect to the norm

$$||x||_{x^*} = x^*(|x|)$$

is an AL — space, say $L_1(x^*)$.

We shall denote by $L_1(\tilde{\mu})$ (respectively $L_1(\mu)$) the space $L_1(x^*)$ constructed as above for $X = \mathcal{E}(\tilde{\mathcal{C}})$ and $x^* = \tilde{\mu}$ (respectively for $X = \mathcal{E}(\mathcal{C}$ and $x^* = \tilde{\mu}|\mathcal{C}$). Then $L_1(\mu)$ can be identified as a closed sublattice of

$L_1(\tilde{\mu})$ and the Lebesgue-Nikodym theorem easily yields the existence of a norm-1 positive projection $P : L_1(\tilde{\mu}) \to L_1(\mu)$.

Notice also that for each $m \in M_E(\tilde{\mu}|\mathcal{C})$ the operator $\Phi_{\mathcal{C}},{}_E(m)$ has a unique extension $T_m \in \mathcal{L}(L_1(\mu), E)$. Then e can be defined as follows:

$$e(m) = \Phi_{\tilde{\mathcal{C}}}^{-1},{}_E (T_m \cdot P| \mathcal{M}(\mathcal{C}))$$

for every $m \in M_E(\tilde{\mu}|\mathcal{C})$, q.e.d.

3.2. **Corollary.** *Let* $\mu : \mathcal{C} \to \mathbf{R}$ a positive measure and let $m \in M_E(\mu)$. *For every Boolean algebra* $\tilde{\mathcal{C}} \supset \mathcal{C}$ *there exists an extension* $\tilde{m} : \tilde{\mathcal{C}} \to E$ *of* m of *finite variation.*

Proof. By Theorem 2.1 above μ has a positive extension $\tilde{\mu}$ to $\tilde{\mathcal{C}}$.

3.3. **Corollary** (see [1]). *Suppose in addition that* $\tilde{\mathcal{C}}$ *is the Boolean* σ*-algebra generated by* $\tilde{\mathcal{C}}$ *and* μ *is* σ*-additive. Then* $\tilde{m}$ *is a* σ*-additive measure that extends m uniquely.*

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